数理解析研究所講究録
912巻 1995年 64-81

A Relationship among Gentzen's Proof-Reduction, Kirby-Paris' Hydra Game, and Buchholz's Hydra Game (Preliminary Report)*

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1 introduction

Kirby-Paris [9] found a certain combinatorial game called Hydra Game whose termination is true but cannot be proved in PA. Cichon [4] gave a new proof based on Wainer's finite characterization of the PA-provably recursive functions by the use of Hardy functions. Both Kirby-Paris and Cichon's proofs on the unprovability result were obtained by a certain investigation on the ordinals less than ϵ_0, the critical ordinal of PA, with the help of the fast and slow growing hierarchies, respectively. On the other hand, Jervell [7] proposed a combinatorial game, called Gentzen Game, on finite binary labeled trees. His Game was defined by abstracting some of the proof reduction procedure of Gentzen's consistency proof of PA [5], which directly implies Gentzen Game's unprovability of PA via Gödel's incompleteness theorem. The rules of Gentzen Game look rather artificial and complicated while those of Kirby-Paris' are more natural and simpler as a combinatorial game. Moreover, the modified proof reduction Jervell considered ignores Gentzen's notion of potential (or height), which makes the Gentzen Game less complicated but the natural termination proof requires much larger than ϵ_0. Hence, the resulting Gentzen Game is much stronger than PA, while Kirby-Paris' Game is considered an optimal game, in the sense that any sub-game restricted to the hydras with an upper bound size turns out to be provable in PA. On the other hand, Jervell's unprovability proof (on Gentzen Game) is direct and clear (thanks to the fact that the Game is directly connected to Gentzen's consistency proof) while the unprovability proofs of Kirby-Paris Game are more involved and complicated. Hence, it is very natural to ask if one can find another way of interpreting Gentzen's proof-reduction procedure into a more natural combinatorial game such as Kirby-Paris', (so that one can get a natural combinatorial game and a direct unprovability proof, at the same time). The purpose of the first part (Section 2) of this paper is to

*This work was partly supported by Grants-in-Aid for Scientific Research of Ministry of Education, Science and Culture of Japan, Keio University Oogata-kenkyuu-jyosei grant, and Mitsubishi Foundation Grant. The first author was also supported by Fellowship for Japanese Junior Scientists from Japan Society for the Promotion of Science.

provide an affirmative answer to this question; We show that Gentzen's proof reduction itself is indeed Kirby-Paris Game under a certain simple and natural interpretation. For that purpose we modify Gentzen's original notion of *potential* or *height* in his consistency proof of PA and introduce a non unique notion of *height*. Then the combinatorial principle Gentzen used implicitly in his proof-reduction for PA becomes exactly Kirby-Paris Game itself. As a direct corollary with the help of Gödel's Incompleteness Theorem Kirby-Paris' unprovability result follows (without using any argument of the fast or slow growing hierarchy). A related result on a logically stronger system, $\Pi_1^1 CA + BI$, in connection with Buchholz's Hydra Game [3] is given in Hamano-Okada [6]. Here, The difference between Buchholz's Hydra Game and Kirby-Paris' lies on the setting where the former uses labeled finite trees while the latter uses unlabeled finite trees. Another striking difference between Buchholz's and Kirby-Paris' is that Buchholz's hydra grows not only in width but also in height while Kirby-Paris' glows only in width. Although the logical strength of Buchholz's Game is much stronger than that of Kirby-Paris' (i.e., ID_ω versus PA), we shall show in Section 3 that when the height-glowing Buchholz's Game on labeled trees is restricted to the one-dimensional case, it is just identified with Kirby-Paris Hydra Game (on two dimensional unlabeled trees), namely the one-dimensional Buchholz's Hydra Game is nothing but the usual Kirby-Paris Game.

2 Kirby-Paris' Hydra Game and Gentzen's Proof Reduction for PA

2.1 Kirby-Paris' Hydra Game on Two-dimensional Finite Unlabeled Trees

In this subsection we shall explain Kirby-Paris' Hydra Game and the combinatorial principle concerning on this game found in [9]. Some derivable rules from Kirby-Paris' Hydra Game are also considered.

Definition 1 *Let $\mathcal{T}$ denote the set of all two-dimensional unlabeled trees. $\mathcal{T}$ is generated inductively as follows.*

- $0 \in \mathcal{T}$
- *If $s \in \mathcal{T}$, then $(s) \in \mathcal{T}$*
- *If $s, t \in \mathcal{T}$, then $s \# t \in \mathcal{T}$*

A *node* of a tree t is defined to be an occurrence of ( ) or 0 in $t \in \mathcal{T}$. A *head* of t is defined to be such a node of the form 0. A *root* of t is defined to be an outer-most occurrence of a node of t. A *connected* tree is 0 or a tree of the form $(t) \in \mathcal{T}$. $\mathcal{T}_{KP}$ denotes the set of all connected trees of $\mathcal{T}$. Each element of $\mathcal{T}_{KP}$ is interpreted as an ordinal less than ϵ_0 inductively as follows.

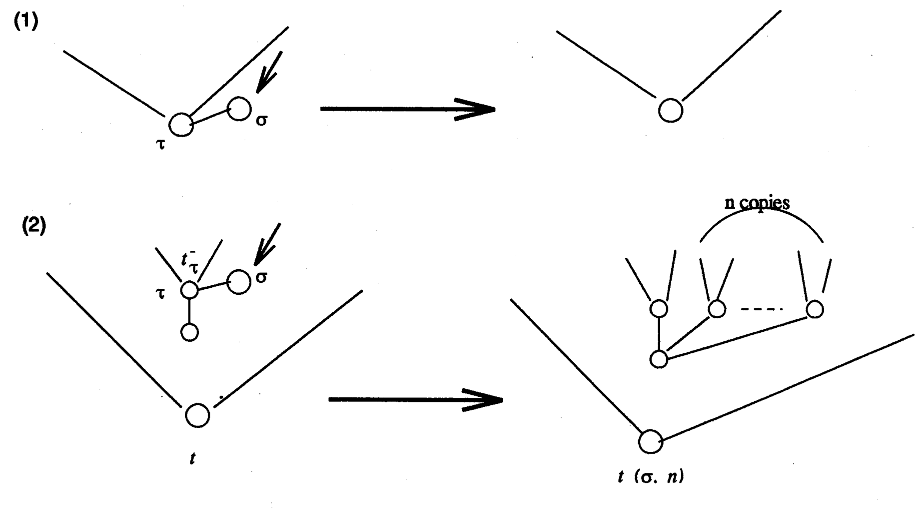
Definition 2 (Ordinal interpretation of a tree in $\mathcal{T}_{KP}$. [9]) *For a connected tree $t \in \mathcal{T}_{KP}$, an ordinal t^* less than ϵ_0 is defined inductively as follows. Let $0, t_1, \ldots, t_n \in \mathcal{T}_{KP}$ where each $t_i (1 \le i \le n)$ is connected .*

- $0^* = 0$.

- $(t_1 \# \cdots \# t_n)^* = \omega^{t_1^*} \# \ldots \# \omega^{t_n^*}$. (# *denotes the natural sum.)*

In this section *a hydra* is defined to be an element of $\mathcal{T}_{KP}$. *Kirby-Paris' Hydra Game* is a battle between Hercules and a hydra. At each stage of a battle Hercules chops off one head of a hydra and chooses a natural number n, then the hydra transforms itself into a new hydra according to the following Rule (Definition 3). Hercules *wins* if after a finite number of stages, nothing is left in the hydra but its root. A *strategy* is a series of Hercules' choice of a head σ to chop off and of a natural number n to choose at each stage: Let $t(\sigma, n)$ denote the transformation of t after Hercules chops off a head σ and chooses a natural number n. A *strategy starting from t* is defined to be a (finite or infinite) sequence of the form $(\sigma_1, n_1) \cdots (\sigma_k, n_k) \cdots$, where each σ_{m+1} is a head of $t(\sigma_1, n_1) \cdots (\sigma_m, n_m)$. A *winning strategy starting from t* is a finite sequence $(\sigma_1, n_1) \cdots (\sigma_m, n_m)$ such that $t(\sigma_1, n_1) \cdots (\sigma_m, n_m) = 0$.

Definition 3 (Rule of Kirby-Paris' Hydra Game [9]) *Let σ denote a head of a hydra t and t^- denote the remaining hydra of t after σ has been chopped off. If Hercules chops off σ and chooses a natural number n, then the hydra transforms itself into a new hydra $t(\sigma, n)$ as follows. Let τ denotes the node immediately below τ. (1) If τ is the root of t , then $t(\sigma, n)$ is t^- (See the figure (1) below). (2) Otherwise $t(\sigma, n)$ results from t^- by sprouting n copies of t_τ^- from the node immediately below τ, where t_τ^- denotes the subtree of t^- determined by τ (See the figure (2) below).*



Proposition 1 (Kirby-Paris [9]) *Every strategy of Kirby-Paris' Hydra Game is a winning strategy.*

Proof.
By one step of Kirby-Paris' Hydra Game the part $((t_1 \# \cdots \# t_k \# 0))$ of a given hydra changes into the part of the form $(\underbrace{(t_1 \# \cdots \# t_k) \# \cdots \# (t_1 \# \cdots \# t_k)}_{n-times})$.

By the correspondence from $\mathcal{T}_{KP}$ into the ordinals less than ϵ_0 defined in Definition 2, the above trees are interpreted as ordinals;

$$((t_1\#\cdots\#t_k\#0))^* = \omega^{\omega^{t_1^*}\#\cdots\#\omega^{t_k^*}\#\omega^0}$$
$$\underbrace{((t_1\#\cdots\#t_k)\#\cdots\#(t_1\#\cdots\#t_k))}_{n-times}{}^* = (\omega^{\omega^{t_1^*}\#\cdots\#\omega^{t_k^*}})\cdot n$$

The corresponding ordinal is strictly decreasing by an application of a rule. Hence well-orderness of the ordinals less than ϵ_0 implies the proposition. □

Let $\prec_{KP}$ denote the reduction relation defined as $\prec_{KP}= \{(t(n),t) \mid 0 \neq t \in \mathcal{T}_{KP}, n \in N\}$. The above proof of Proposition 1 implies that $\prec_{KP}$ is contained in the ordinals less than ϵ_0.

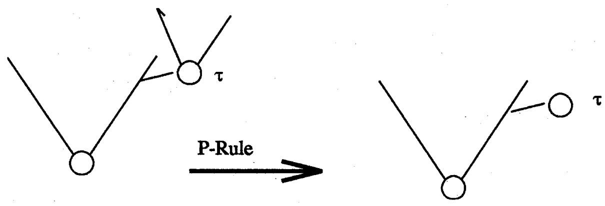
Consider the subset $\mathcal{T}(n)$ of $\mathcal{T}$ defined as $\mathcal{T}(n) = \{t \in \mathcal{T} \mid$ No node is more than n segments away from the root below it.$\}$. Let $KP(n)$ denote the statement "For every $t \in \mathcal{T}(n)$ every recursive strategy starting from t is a winning strategy.". From the fact that the transfinite induction up to each ordinal less than ϵ_0 is provable in PA, we have the next proposition.

Proposition 2 (Kirby-Paris[9]) *For an arbitrary natural number n,*

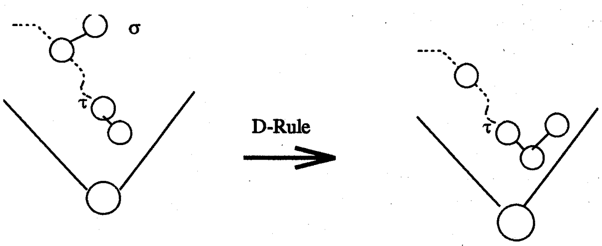
$$PA \vdash KP(n)$$

For convenience in the next section we shall prepare P-Rule and D-Rule which are derivable by a finite number of steps of Kirby-Paris' Hydra Game. We use the name KP-Rule to denote the basic Rule of Kirby-Paris' Game defined in Definition 3.

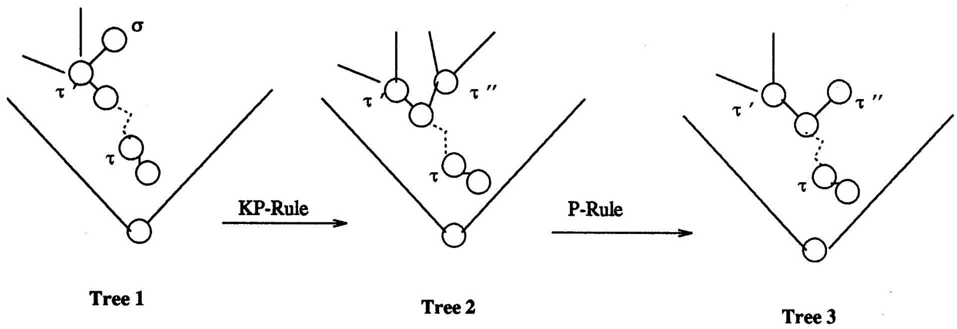
P-Rule (Pruning Rule) Let τ be an arbitrary node (not necessarily a head) of a hydra t. t transforms itself into such a hydra that is obtained from t by pruning all the nodes of t_τ (the subtree of t determined by τ) except its root (i.e.τ).



D-Rule (Descending Rule) Let σ be a head of a hydra t and τ a node below σ. t transforms itself into such a hydra that is obtained from t^- (the resulting hydra after σ has been chopped off) by attaching a copy of σ to the node immediately below τ.



Derivability of P-Rule from KP-Rule is trivial from the following: For a given hydra t, let m denote the number of nodes consisting of this hydra. If Hercules chooses such a strategy that chooses a natural number 0 at every stage then he wins after $m+1$ stages against t. Derivability of D-Rule from KP-Rule is sketched as following figure. From KP-Rule Tree 1 transforms itself into Tree 2. From P-Rule Tree 2 transforms itself into Tree 3. Therefore we can prove the derivability by induction on the number of occurrences of the nodes between σ and τ.



2.2 Direct Interpretation of Gentzen's Proof Reductions by Means of Kirby-Paris' Hydra Game

In this subsection we shall show that Gentzen's proof reduction for PA [5] is directly expressive by reduction relation $\prec_{KP}$ (Theorem 1). As a corollary of Theorem 1 with the help of Gödels' Incompleteness Theorem, Kirby-Paris' independence result follows (Proposition 3). In this section by PA we mean Peano Arithmetic in Takeuti [12]. We assume that the readers are familiar with the notions used in Chapter 2 of Takeuti. [12].

Definition 4 [1] *A proof P in PA is called a proof with* height *denoted by $< P, h >$ if for each sequent S of P a natural number $h(S)$ satisfying the following condition is assigned. Here $g(D)$ is* grade *of a formula D.*

1. *$h(S) = 0$ if S is the end sequent of P.*
2. *$h(S) \geq h(S')$ if S is an upper sequent of an inference except cut and ind where S' is the lower sequent of the inference.*
3. *$h(S) \geq max\{h(S'), g(D)\}$ if S is an upper sequent of cut, where D is the cut formula of the inference, and S' is the lower sequent of the inference.*
4. *$h(S) \geq max\{h(S'), g(D)\} + 1$ if S is an upper sequent of an ind, where D is the induction formula of the inference, and S' is the lower sequent of the inference.*

Definition 5 *For $t \in \mathcal{T}$, $(t)^n$ is defined by induction on n as follows.*

[1] Our assignment of *height* is a slight modification of that in Definition 12.4 [12]. Our assignment of *height* to each sequence is not uniquely determined. This slight modification becomes essential in interpreting Gentzen's Proof Reduction by means of Kirby-Paris' Hydra Game in (Case 1.2), (Case 2), (Preparation 2) of Theorem 1.

- $(t)^0 = t$.
- $(t)^{n+1} = (0\#(t)^n)$.

Definition 6 (Tree interpretation of a proof in PA) *Let $< P, h >$ be a proof with* height *h in PA. Each occurrence of a sequent S in $< P, h >$ is interpreted as $T(S) \in \mathcal{T}$ inductively as follows. Then we shall define $T(< P, h >) \in \mathcal{T}_{KP}$ as $T(< P, h >) = (T(S))$ where S is the end-sequent of $< P, h >$.*

0. $T(S) = 0$ if S is an initial sequent of $< P, h >$.

Suppose that tree-interpretation $T(S')$ and $T(S'')$ of S' and S" have been assigned.

$$\frac{S' \qquad (S'')}{S} J$$

where S'' does not exist when J has a unique upper sequent. Then T(S) is determined as follows.

1. *If J is a weak structural inference, then $T(S) = (T(S'))^{h(S')-h(S)}$*
2. *If J is $\neg$, $\wedge; left$, $\vee; right$, or first order $\forall$, then $T(S) = (0\#T(S'))^{h(S')-h(S)}$*
3. *If J is $\wedge; right$, or $\vee; left$, then $T(S) = (T(S')\#T(S''))^{h(S')-h(S)}$*
4. *If J is an induction, then $T(S) = (0\#T(S'))^{h(S')-h(S)}$*
5. *If J is a cut, then $T(S) = (T(S')\#T(S''))^{h(S')-h(S)}$*

By the above interpretation a proof with *height* is interpreted into $\mathcal{T}_{KP}$. The next theorem asserts that $<\prec_{KP}, \mathcal{T}_{KP}>$ is strong enough to prove the consistency of PA by a finitary method.

Theorem 1 *Each step of Gentzen's proof reduction for PA is interpretable by a finite number of steps of Kirby-Paris' Hydra Game.*

Proof.
By *Gentzen's proof reduction for PA* we mean Takeuti's refinement of it described in Lemma 12.8 (p 105 - p 114) in [12] .[2] Given a proof $< P, h >$ with *height* of PA, let $< P', h' >$ denote the proof which results from $< P, h >$ through one step of proof reduction described in Lemma 12.8 (p 105 - p 114) [12].
We shall add the redundant inference; called term-replacement.

$$\frac{\Gamma_1, F(s), \Gamma_2 \rightarrow \Delta}{\Gamma_1, F(t), \Gamma_2 \rightarrow \Delta}$$

[2]Here in this section we do not mention explicitly the ordinal assignment to a sequent of a proof but only mention tree interpretation of it. But note that our ordinal assignment introduced with the bypass of tree-interpretation of a sequent (Definition 2) becomes slightly different from Takeuti's in Definition 12.6 [12].

where s and t are closed terms expressing the same number.
Our definition of the assignment of height to the upper sequent of term-replacement follows Definition 4 (2), and that of the ordinal assignment of the lower sequent of term-replacement follows Definition 6 (1).
To obtain the theorem, it suffices to prove Lemma 1.

Lemma 1 *Let* (a) *and* (b) *are following assertions.*

(a) *The tree interpretation of* $< P', h' >$ *remains the same as that of* $< P, h >$.

(b) *The tree interpretation of* $< P', h' >$ *is derivable from that of* $< P, h >$ *by making a finite number of uses of KP-Rule, P-Rule and D-Rule.*

Then the following holds.

1. *In each of the following (Preparation 0)* $\sim$ *(Preparation 2), either (a) or (b) holds.*

2. *In each of the following (Case 1)* $\sim$ *(Case 4), (b) holds.*

Proof of Lemma 1.
(Preparation 0) Suppose that the end-piece of P contains a free variable which is not used as an eigenvariable. Then replace this redundant free variable by the constant 0. Obviously by this substitution (a) holds.
(Case 1) Suppose that P contains an application of induction in its end-piece. Let I be a lower most induction in the end-piece of P.
$< P, h >$ is of the following form. Here l and $m+1$ denote the height of $A(0), \Gamma \rightarrow \Delta, A(t)$ and of $A(a), \Gamma \rightarrow \Delta, A(a')$ respectively with $l \leq m$.

$$\begin{array}{cl}
\vdots\; Q(a) & \\
\hline
A(a), \Gamma \rightarrow \Delta, A(a') \; J & \\
\hline
A(0), \Gamma \rightarrow \Delta, A(t) \; I & \\
\vdots & \\
\rightarrow &
\end{array}
\qquad
\begin{array}{c} k+1 \\ m+1 \\ l \end{array}$$

(Case 1.1) The case where $t = 0$. $< P', h' >$ is obtained from replacing the part of $< P, h >$ above $A(0), \Gamma \rightarrow \Delta, A(t)$ by

$$\cfrac{\cfrac{\underline{A(0) \rightarrow A(0)}}{A(0), \Gamma \rightarrow \Delta, A(0)}}{A(0), \Gamma \rightarrow \Delta, A(t)}\; term - replacement\; .$$

In this case the tree interpretation from $< P, h >$ to $< P', h' >$ corresponds exactly to our P-Rule, hence (b) holds.
(Case 1.2) The case where $t \neq 0$. $< P', h' >$ is obtained by eliminating J by the following, where t=n for some numeral n except 0.

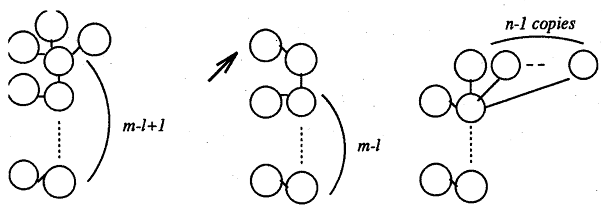
$$\cfrac{\cfrac{\vdots\; Q(0)}{A(0),\Gamma \to \Delta, A(0')}\,J_0 \qquad \cfrac{\vdots\; Q(0')}{A(0'),\Gamma \to \Delta, A(0'')}\,J_1 \qquad \begin{matrix} k+1 \\ m \end{matrix}}{\cfrac{A(0),\Gamma,\Gamma \to \Delta,\Delta, A(0'')}{A(0),\Gamma \to \Delta, A(0'')}}\; m$$

$$\ddots$$

$$\cfrac{\cfrac{\cfrac{A(0),\Gamma \overset{\vdots}{\to} \Delta, A(n-1) \qquad \cfrac{\vdots\; Q(n-1)}{A(n-1),\Gamma \to \Delta, A(n)}\,J_{n-1}}{A(0),\Gamma,\Gamma \to \Delta,\Delta, A(n)}\; \begin{matrix} m \\ l \end{matrix}}{A(0),\Gamma \to \Delta, A(n)}}{\begin{matrix} A(0),\Gamma \to \Delta, A(t) \\ \vdots \\ \to \end{matrix}}\; term-replacement$$

Our tree interpretation from $< P, h >$ to $< P', h' >$ is divided into the following two cases:

(Case 1.2.1) The case where $A(a), \Gamma \to \Delta, A(a')$ is an initial sequent of $< P, h >$. $< P, h >$ is interpreted as Tree 1. By pruning two top nodes, Tree 1 transforms itself into Tree 2. By chopping off the top node marked with $\nearrow$, Tree 2 transforms itself into Tree 3, which is the interpretation of $< P', h' >$. Hence (b) holds.



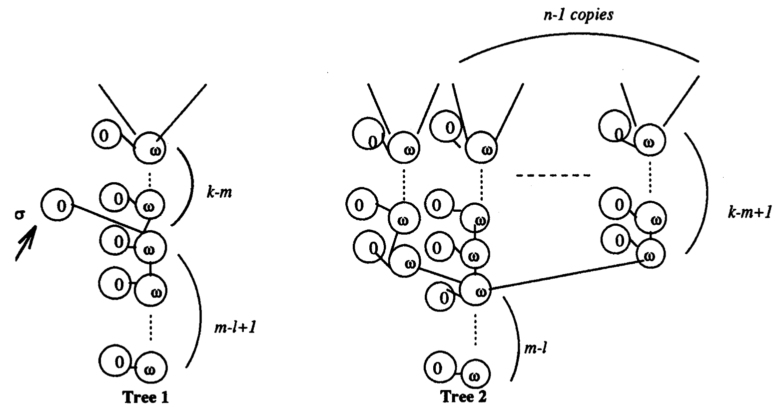
Tree 1 Tree 2 Tree 3

(Case 1.2.2) The case where $A(a), \Gamma \to \Delta, A(a')$ is not an initial sequent of $< P, h >$. Let J be the inference in P whose lower sequent is $A(a), \Gamma \to \Delta, A(a')$. From the definition of *proof with height* (Definition 4), the height of an upper sequent of J in P must be $k+1$ with $m \leq k$. In $< P', h' >$, let J_i be a copy of J in $Q(i)$, and S_i be an upper sequent of J_i where $0 \leq i < n$. Therefore we shall define height h' in $< P', h' >$ as

$$\begin{cases} h'(A(i), \Gamma \to \Delta, A(i')) = m \\ h'(S_i) = k+1 \end{cases}$$

(cf. Definition 4). By virtue of this height h' of $< P', h' >$, we shall carry out the tree transformation as follows.

$< P, h >$ is interpreted as Tree 1. Let σ be the top node corresponding to the indicated 0 of the tree interpretation of $A(0), \Gamma \to \Delta, A(t)$ (cf. Definition 6 (4)). By chopping off σ, we obtain Tree 2, which is exactly the interpretation of $< P', h' >$. because of h' described above. Hence (b) holds.



(Preparation 1) Suppose that the end-piece of P contains an equality axiom of the form $s = t, A(s) \to A(t)$ as an initial sequent with the height h. Let m and n denote numerals equal to s and t respectively.

If $m = n \to$ is an axiom, then $s = t, A(s) \to A(t)$ is replaced by

$$\begin{array}{cc} \underline{\quad m = n \to \quad} & h \\ \overline{m = n, A(m) \to A(n)} & \vdots \\ \underline{term - replacements} & \vdots \\ s = t, A(s) \to A(t) & h \end{array}.$$

By virtue of the assignment of tree interpretation regarding with the term-replacement defined in (Preparation 0), the corresponding tree interpretation of $s = t, A(s) \to A(t)$ remains the same through this modification; it remains 0. Hence (a) holds. It is treated in the same way as when $\to m = n$ is an axiom.

(Case 2) The case where the end-piece of P contains a logical initial sequent.
$< P, h >$ is of the following form.

$$\dfrac{\begin{array}{cc} & D \to D \\ \vdots & \vdots \\ \Gamma \to \Delta, \tilde{D} & \tilde{D}, \Pi \to \Lambda_1, \tilde{D}, \Lambda_2 \end{array}}{\begin{array}{c} \Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\ \vdots \\ \to \end{array}} J \quad \begin{array}{c} m \\ l \end{array}$$

$< P', h' >$ is as follows.

$$\dfrac{\begin{array}{c} \vdots \\ \underline{\Gamma \to \Delta, \tilde{D}} \end{array}}{\begin{array}{c} \Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\ \vdots \\ \to \end{array}} \quad \begin{array}{c} m \\ l \end{array}$$

Note that our definition of height allows such an assignment h' in P' that h' is the restriction of h in P with respect to P' (cf. Definition 4). Therefore the tree which represents $< P', h' >$ is strictly a part of the tree which represents $< P, h >$. Consequently this proof reduction from $< P, h >$ to $< P', h' >$ corresponds exactly to our P-Rule. (Prune the forest corresponding to the tree interpretation of $\tilde{D}, \Pi \to \Lambda_1, \tilde{D}, \Lambda_2$ in $< P, h >$.) Hence (b) holds.

(Preparation 2) The case where the end-piece of P contains a weakening. Let I be a lower most weakening inference in the end-piece. Since the end-sequent is empty, there must exist a cut J below I, whose cut formula is a descendant of the principal formula of I: $< P, h >$ is of the following form.

$$\begin{array}{ccc}
 & \vdots & \\
 & \dfrac{\Pi_1, \to \Lambda_1}{D, \Pi_1 \to \Lambda_1}\, I & \begin{array}{c} n \\ m \end{array} \\
\vdots & \vdots & \\
\multicolumn{2}{c}{\dfrac{\Gamma \to \Delta, \tilde{D} \qquad \tilde{D}, \Pi \to \Delta}{\Gamma, \Pi \to \Delta, \Lambda}\, J} & \begin{array}{c} l \\ k \end{array} \\
\multicolumn{2}{c}{\vdots} & \\
\multicolumn{2}{c}{\to} &
\end{array}$$

(Preparation 2.1) The case where no contraction is applied to a descendant of D from the inference I through J. By regulating some exchanges from P, $< P', h' >$ is obtained as follows.

$$\begin{array}{cc}
\vdots & \\
\dfrac{\Pi_1 \to \Lambda_1}{\overline{\Pi_1 \to \Lambda_1}} & \begin{array}{c} n \\ m \end{array} \\
\vdots & \\
\dfrac{\Pi \to \Delta}{\overline{\Gamma, \Pi \to \Delta, \Lambda}} & \begin{array}{c} l \\ k \end{array} \\
\vdots & \\
\to &
\end{array}$$

In this case, in the same manner as in Case 2, the tree interpretation from $< P, h >$ to $< P', h' >$ corresponds exactly to our P-Rule. Hence (b) holds.

(Preparation 2.2) If it is not the case of Preparation 2.1. Let $\tilde{I}$ be the uppermost contraction applied to a descendant of D. $< P', h' >$ is obtained by replacing the part of $< P, h >$ above $\tilde{D}, \Pi \to \Lambda$ as follows.

$$\begin{array}{ccc}
< P, h > & & < P', h' > \\
\vdots & & \vdots \\
\dfrac{\Pi_1 \to \Lambda_1}{D, \Pi_1 \to \Lambda_1}\, I & \begin{array}{c} n \\ m \end{array} & \dfrac{\Pi_1 \to \Lambda_1}{\overline{\Pi_1 \to \Lambda_1}}\; \begin{array}{c} n \\ m \end{array} \\
\vdots & & \vdots \\
\dfrac{D', D', \Pi_1' \to \Lambda_1'}{D', \Pi_1' \to \Lambda_1'}\, \tilde{I} & \begin{array}{c} n' \\ m' \end{array} & \dfrac{D', \Pi_1' \to \Lambda_1'}{\overline{D', \Pi_1' \to \Lambda_1'}}\; \begin{array}{c} n' \\ m' \end{array} \\
\vdots & & \vdots \\
\tilde{D}, \Pi \to \Lambda & & \tilde{D}, \Pi \to \Lambda
\end{array}$$

Through the above modification the corresponding tree interpretation of $\tilde{D}, \Pi \to \Lambda$ remains the same; the tree interpretation of $< P', h' >$ is completely the same as that of

$< P, h >$. Hence (a) holds.
(Case 1), (Preparation 1), (Case 2), and (Preparation 2) having been in hand, we can assume that there exists a suitable cut in the end-piece of P (cf. Sublemma 12.9 in Takeuti [12]). Now let J be a suitable cut of $< P, h >$. The essential proof reductions and tree interpretations are divided into the following two cases (Case 3 and Case 4) according to the outermost logical symbol of the cut formula of J.
(Case 3) The case where the cut formula of J is of the form $A \wedge B$, $\neg A$, $A \vee B$, or $\forall x A(x)$. Each proof reduction and tree interpretation are to be treated similarly. Without loss of generality, we consider the case $A \wedge B$.
$< P, h >$ is of the following form.

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \to \Delta_1, A_1\end{array}\quad\begin{array}{c}\vdots\\ \Gamma_2 \to \Delta_2, B_1\end{array}}{\Gamma_1,\Gamma_2 \to \Delta_1,\Delta_2,A_1\wedge B_1}\begin{array}{l}k_1\\ h_1\end{array}\\ \vdots \\ \Gamma_3 \to \Delta_3, A\wedge B\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ A_2,\Pi_2 \to \Lambda_2\end{array}}{A_2\wedge B_2,\Pi_2\to\Lambda_2}\begin{array}{l}k_2\\ h_2\end{array}\\ \vdots\\ A\wedge B,\Pi_3\to\Lambda_3\end{array}}{\Gamma_3,\Pi_3\to\Delta_3,\Lambda_3}\\
\vdots\\
\dfrac{}{\Phi\to\Psi}\begin{array}{l}l\\ n\end{array}\\
\vdots\\
\to
\end{array}
$$

$\Phi \to \Psi$ is upper most sequent below J
whose height n is less than l, here l denotes the height of
an upper sequent of J.

Then, $< P', h' >$ is the following:

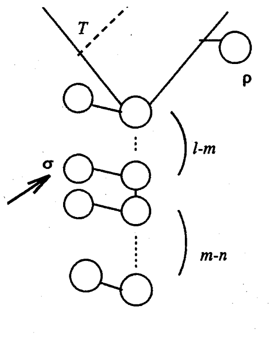
$$
\begin{array}{c}
\dfrac{\begin{array}{c}\dfrac{\underline{\begin{array}{c}\vdots\\ \Gamma_1 \to \Delta_1, A_1\end{array}}}{\Gamma_1,\Gamma_2 \to A_1,\Delta_1,\Delta_2,A_1\wedge B_1}\begin{array}{l}k_1\\ h_1\end{array}\\ \vdots\\ \Gamma_3 \to A_1,\Delta_3, A\wedge B\end{array}\quad \begin{array}{c}\vdots\\ A\wedge B,\Pi_3\to\Lambda_3\end{array}}{\Gamma_3,\Delta_3 \to A,\Delta_3,\Lambda_3}
\qquad
\dfrac{\begin{array}{c}\vdots\\ \Gamma_3 \to \Delta_3, A\wedge B\end{array}\quad \begin{array}{c}\dfrac{\underline{\begin{array}{c}\vdots\\ A_2,\Pi_2 \to \Lambda_2\end{array}}}{A_2\wedge B_2,\Pi_2,A_2\to\Lambda_2}\begin{array}{l}k_2\\ h_2\end{array}\\ \vdots\\ A\wedge B,\Pi_3,A\to\Lambda_3\end{array}}{\Gamma_3,\Pi_3 \to A,\Delta_3,\Lambda_3}
\end{array}
$$

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\vdots}{\Phi\to A,\Psi}}{\Phi\to\Psi,A}\quad\dfrac{\dfrac{\vdots}{\Phi,A\to\Psi}}{A,\Phi\to\Psi}\begin{array}{l}l\\ m\\ m\\ n\end{array}}{\dfrac{\Phi,\Phi\to\Psi,\Psi}{\Phi\to\Psi}}\\
\vdots\\
\to
\end{array}
$$

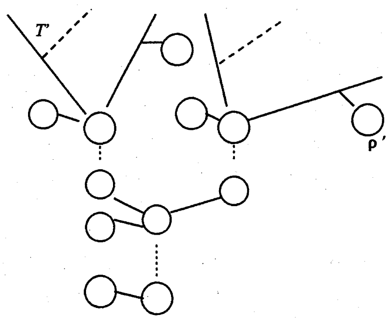
Let h' be the height of P'. Then as denoted in the above proof figure, h' is as follows.

- $h'(\Phi \to \Psi, A) = h'(A, \Phi \to \Psi) = m$ with $n \leq m < l$.
- If S is a sequent of $< P', d' >$ except $\Phi \to \Psi, A$ and $A, \Phi \to \Psi$, then in $< P, h >$ there exists a sequent $\tilde{S}$ of which S is a copy, and $h'(S) = h(\tilde{S})$.

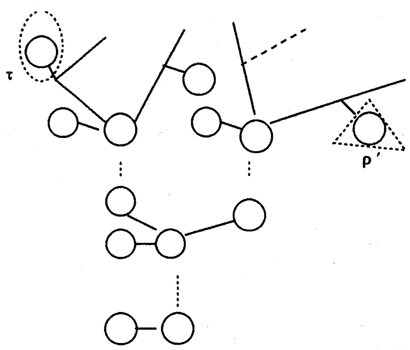
By virtue of this height assignment of $< P', h' >$, we interprete the tree transformation from $< P, h >$ to $< P', h' >$ as follows: We interpret $< P, h >$ as Tree 1, where T is the element of $\mathcal{T}$ corresponding to the sequent $\Gamma_2 \to \Delta_2, B_1$, and ρ is the head corresponding to the indicated 0 of the tree interpretation of $A_2 \wedge B_2, \Pi_2 \to \Lambda_2$ (cf. Definition 6 (2)). By chopping off the node denoted by σ, Tree 1 transforms itself into Tree 2 . In Tree 2, let T' be the left-hand side copy of T, and ρ' the right-hand side copy of ρ. By P-Rule, pruning all the nodes except one of the lowest nodes τ of T', we obtain Tree 3. By D-Rule, we shift τ and ρ' down respectively to the nodes where σ used to be attached. This completes our tree transformation; Tree 4 is exactly the interpretation of $< P', h' >$ because of the assignment of h' described above. Hence (b) holds.



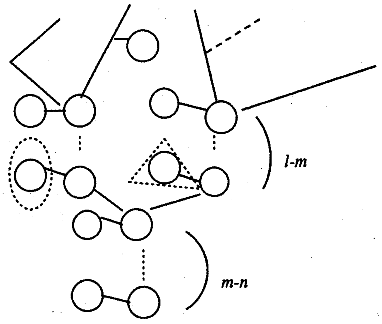
Tree 1



Tree 2



Tree 3



Tree 4

□

From Theorem 1, there exists at least one strategy of Kirby-Paris' Hydra Game whose termination guarantees the termination of Gentzen's proof reduction on his consistency proof of PA [5, 12]. So by virtue of Gödel's Incompleteness Theorem, the statement *'the strategy of such a kind is a winning strategy.'* is not provable in PA. Now we reach directly Kirby-Paris' independence result as follows.

Proposition 3 (Kirby-Paris[9]) *The statement "For every hydra $t \in \mathcal{T}_{KP}$ every recursive strategy starting from t is a winning strategy" is not provable in PA.*

3 A Relationship between Kirby-Paris' Hydra Game and Buchholz's Hydra Game.

Buchholz extended Kirby-Paris' Hydra Game significantly by defining a game on labeled finite trees in which a hydra grew not only in width but also in height (cf. Buchholz [3], Hamano-Okada [6] for the definition of the general case of Buchholz's Hydra Game). In this section we shall restrict our attention to the one-demensional version of Buchholz's Hydra Game, which we shall call *one-dimensional Buchholz's Hydra Game* and denote $B^1_{\omega+1}$-Game. One dimensional labeled tree is a finite sequence of elements from $\omega+1=\{0,1,\ldots,\omega\}$. $\mathcal{S}_{\omega+1}$ denotes the set of all one-dimensional trees. The *root* and the *head* of an one-dimensional tree is defined to be respectively the beginning and the ending element of the sequence. An one-dimensional labeled *hydra* is an one-dimensional labeled tree whose root is 0. We use the notation $\mathcal{H}^1_{\omega+1}$ to denote the set of all one-dimensional hydras labeled from $\omega+1=\{0,1,\ldots,\omega\}$. One-dimensional version of Buchholz's Hydra Game, denoted by $B^1_{\omega+1}$*-Game*, is a battle between Hercules and one-dimensional hydra. At each step of a battle Hercules chops off the head of a given hydra S, and chooses a natural number n, then hydra S transforms itself into a new hydra $S(n)$ in three different ways according to the labels of the head of the hydra following three rules below. For an one-dimensional labeled tree S, we use the notation S^- and S_i to denote respectively the subsequences of S obtained by deleting its head and by deleting i-many nodes from the root.

Definition 7 (One-dimensional version of Buchholz's Hydra Game) [3]
Rules of $B^1_{\omega+1}$-Game are the following three kinds of rules to transform a hydra S into $S(n)$, for an arbitrary chosen natural number n.

Rule 1 *The case where the head of a hydra is* 0.
$S(n)$ is obtained from S by deleting its head (i.e.0).

Rule 2 *The case where the head of the hydra is* $m+1$.
Let a_j be the first occurrence of an element from the head of S such that $a_j \leq m$. $S(n)$ is the sequence $S^- \underbrace{m\, S_j^- \cdots m\, S_j^-}_{n-times}\ 0$.
For example if S is $0,1,2,2$ then $S(n)$ is $0,1,2,\underbrace{1,2\cdots1,2}_{n-times},0$

Rule 3 *The case where the head of a hydra is* ω.
$S(n)$ is obtained from H by simply changing the head by $n+1$.

Let $\prec_{B^1_{\omega+1}}$ denote the reduction relation defined as $\prec_{B^1_{\omega+1}}=\{(S(n),S)\mid 0\neq S\in\mathcal{H}^1_{\omega+1}, n\in N\}$. Then it is known that $\prec_{B^1_{\omega+1}}$ is well founded (cf. [3] [11]). I.e, every recursive strategy of $B^1_{\omega+1}$-Game is a winning strategy.

Let $\mathcal{T}_2$ denote the set of all two dimensional forests each node of which is labeled from $\{0,1\}$. Elements of $\mathcal{T}_2$ corresponds one to one into the elements of $\mathcal{H}^1_{\omega+1}$ as follows. We

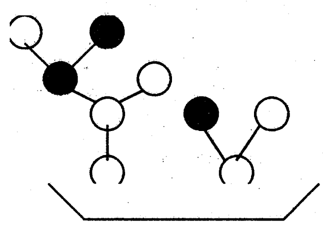
[3]The definition of the following rules are almost the same as those of Buchholz's Game [3] restricted to one-dimension except that Rule 2 is the restriction of such the extension of Case 2 of [3] as a hydra grews up in height also while making n-copies.

use a *white* node to describe a node labeled with 0 and a *black* node to describe a node labeled with 1.

Definition 8 (Interpretation of $\mathcal{T}_2$ into $\mathcal{H}^1_{\omega+1}$) • *The tree consisting of one white node is interpreted as* 0.

- *The tree consisting of one black node is interpreted as* $0\,\omega$.
- *If a forest* T *consists of trees* $T_1, T_2, \ldots, T_n$*, and each* T_i *is interpreted as a sequence* S_i*, then* T *is interpreted as sequence* $S_1\, S_2 \cdots\, S_n$.
- *If a forest* T *is a rooted tree of the form* $\overset{T_1}{\overset{|}{\circ}}$ ($\overset{T_1}{\overset{|}{\bullet}}$*, respectively), and* T_1 *is interpreted as a sequence* S_1*, then* T *is interpreted as a sequence* $0\,\mathtt{Suc}(S_1)$ ($0\,\omega\,\mathtt{Suc}(S_1)$*, respectively). Here, for a sequence* S*,* $\mathtt{Suc}(S)$ *denotes the sequence resulting from* S *by replacing each occurrence of a natural number by its successor. Note that an occurrence of* ω *in* S *remains the same in* $\mathtt{Suc}(S)$.

Example 1 *The following forest corresponds to a hydra of the form* $0, 1, 2, \omega, 3, 3, \omega, 2, 0, 1, \omega, 1$.



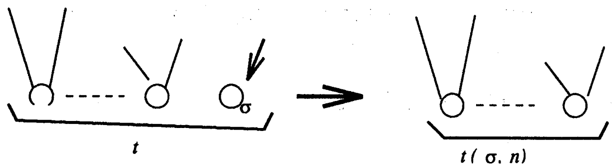
Forest 1

Let $\mathcal{H}^1_\omega$ denote such a subset of $\mathcal{H}^1_{\omega+1}$ that consists of all one-dimensional hydras labeled from $\omega = \{0, 1, \ldots\}$. As a restricted case of Definition 8, the elements of $\mathcal{T}$ corresponds one to one into the elements of $\mathcal{H}^1_\omega$. Let B^1_ω-Game be the same Game as Definition 7 with the restriction that the hydras are only the elements from $\mathcal{H}^1_\omega$; Rule 3 is not necessary in B^1_ω-Game. And $\prec_{B^1_\omega}$ denotes the reduction relations defined as $\prec_{B^1_\omega} = \{(S(n), S) \mid 0 \neq S \in \mathcal{H}^1_\omega, n \in N\}$.

In observing a relationship with B^1_ω-Game it is natural to slightly extend Kirby-Paris' Hydra Game over $\mathcal{T}$, i.e., the forests rather than the trees.

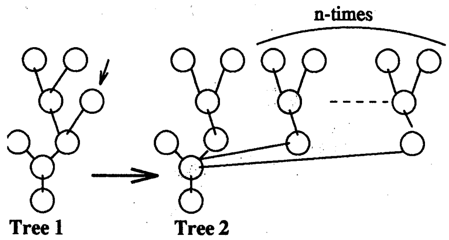
Definition 9 (Extended Kirby-Paris' Hydra Game [11]) *Extended Kirby-Paris' Hydra Game is the same as Definition 3 except that* a hydra t *denotes not only an element of* $\mathcal{T}_{KP}$ *but also an element of* $\mathcal{T}$*, and that we add the following case (3) in Definition 3, in order to deal with a forest.*
(3) If σ *is a root of* t*, then* $t(\sigma, n)$ *is* t^- *(see the figure below).*

In the same manner as in Definition 2, an element from $\mathcal{T}$ is interpreted as an ordinal less than ϵ_0, hence parallel arguments to the proofs of Proposition 1 and Proposition 2 also work. Let $\prec_{EKP}$ denote the reduction relation defined as $\prec_{EKP} = \{(t(n), t) \mid 0 \neq t \in \mathcal{T}, n \in N\}$

(3)


Lemma 2 *One step of Extended Kirby-Paris' Hydra Game corresponds exactly to the one or two steps of B^1_ω-Game; $\prec_{EKP}$ is contained in $\prec_{B^1_\omega}$.*

Sketch of proof. One can check easily that for example, the following one step transformation from Tree 1 to Tree 2 in Kirby-Paris' Hydra Game corresponds exactly to Rule 2 and Rule 1 in succession in B^1_ω-Game: Tree 1 is interpreted as $0,1,2,2,3,4,4,3 \in \mathcal{H}^1_\omega$. $0,1,2,2,3,4,4,3$ transforms itself into $0,1,2,2,3,4,4,\underbrace{2,3,4,4,\ldots 2,3,4,4}_{n-times},0$ by Rule 2. Then by Rule 1 this hydra transforms itself into $0,1,2,2,3,4,4,\underbrace{2,3,4,4,\ldots 2,3,4,4}_{n-times}$.



□

The above interpretation indicates that Kirby-Paris' Hydra Game is embedded into B^1_ω-Game. From Theorem 1, the next theorem follows.

Theorem 2 *The statement " For every $S \in \mathcal{H}^1_\omega$, every recursive strategy starting from S is a winning strategy in B^1_ω-Game" is not provable in PA.*

Let $\mathcal{S}_\omega$ denote the subset of $\mathcal{S}_{\omega+1}$ that consists of all trees labeled from ω. Let a_n denote a node occuring in $S \in \mathcal{S}_\omega$. By a *node immediately below* a_n we mean such a node a_m that $a_m < a_n$ holds with $m < n$ and that $a_i \geq a_n$ holds for all i with $m < i < n$,

By a *node immediately above* a_n we mean such a node a_k that a_n is the node immediately beloew a_k.

Definition 10 (Interpretation of $\mathcal{H}^1_\omega$ onto $\mathcal{T}$) *Let $S = a_1, \ldots, a_n \in \mathcal{S}_\omega$. Then $\overline{S} \in \mathcal{S}_\omega$ is defined so that $S \in \mathcal{H}^1_\omega$ implies $\overline{S} \in \mathcal{T}$ by the induction on n as follows.*

1 *If $n = 1$, then $\overline{S} = S$.*

2 *If $S = S', a_{n+1}$ with $S' = a_1, \ldots, a_n$, then $\overline{S}$ is obtained from $\overline{S'}$ as follows.*

2.1 *If $a_n \geq a_{n+1}$, then $\overline{S} = \overline{S'}, a_{n+1}$.*

2.2 *If $a_n < a_{n+1}$, then $\overline{S} = \overline{S'}, a_n + 1, \ldots, a_{n+1} - 1, a_{n+1}$.*

Example 2 *If $S = 0,3,3,5,2 \in \mathcal{H}^1_\omega$, then $\overline{S} = 0,1,2,3,3,4,5,2 \in \mathcal{T}$.*

Lemma 3 *Each step of B^1_ω-Game corresponds to exactly one step of Kirby-Paris' Hydra Game under the interpretation defined in Definition 10.*

Proof. In the case where a given hydra has the head 0 the theorem is obvious; Rule 1 of Definition 7 corresponds exactly to (3) of Definition 9.
Let us consider the case where a given hydra has the head $i+1$; the hydra is of the form $S = S'\ j\ H\ i+1 \in \mathcal{H}^1_\omega$, where the indicated j is the node immediately below the indicated head $i+1$. In B^1_ω-Game, one step reduction against S means one step of Rule 2 to be necessarily followed by one step of Rule 1 described as follows.

$$S'\ j\ H\ i+1 <_{B^1_\omega} S'\ j\ H\ \underbrace{i\ H \cdots i\ H}_{n-times}\ 0 <_{B^1_\omega} S'\ j\ H\ \underbrace{i\ H \cdots i\ H}_{n-times}$$

On the other hand

$$\begin{aligned} \overline{S'\ j\ H\ i+1} &= \overline{S'\ j\ H}\ i+1 \\ \overline{S'\ j\ H\ \underbrace{i\ H \cdots i\ H}_{n-times}} &= \overline{S'\ j\ H}\ \underbrace{\overline{i\ H} \cdots \overline{i\ H}}_{n-times}. \end{aligned}$$

Obviously $\overline{i\ H}$ is the subtree of $\overline{S}$ determined by the node immediately below its head $i+1$. Hence $\overline{S'\ j\ H\ i+1} <_{KP} \overline{S'\ j\ H\ \underbrace{i\ H \cdots i\ H}_{n-times}}$
□
The above lemma implies the following lemma.

Lemma 4 *$\prec_{EKP}$ is strong enough to prove the well orderness of $\prec_{B^1_\omega}$.*

Let $<^+_{B^1_\omega}$ denote the transitive closure of $<_{B^1_\omega}$. The following is the Corollary of Lemma 2.

Corollary 1 *If $\overline{S_1} = \overline{S_2}$ with $S_1, S_2 \in \mathcal{H}^1_\omega$, then neither $S_1 <^+_{B^1_\omega} S_2$ nor $S_2 <^+_{B^1_\omega} S_1$ holds.*

Proof. From the proof of Lemma 2, $S_1 <_{B^1_\omega} S_2$ implies $\overline{S_1} <_{KP} \overline{S_2}$. Hence the assertion holds. □
Consider the subset $\mathcal{S}_\omega(n)$ of $\mathcal{S}_\omega$ as $\mathcal{S}_\omega(n) = \{S \in \mathcal{S}_\omega \mid \text{No node of } S \text{ is more than } n.\}$. It is easily checked that $\mathcal{T}(n)$ is embeded into $\mathcal{S}_\omega(n)$ under the interpretation of Lemma 2 and that if $S \in \mathcal{S}_\omega(n)$, then $\overline{S} \in \mathcal{T}(n)$. Consider the statement $B^1_\omega(n)$ for an fixed natural number n to denote "For every $S \in \mathcal{S}_\omega(n)$ every recursive strategy starting from S is a winning strategy.".

Lemma 5 *$KP(n) \Leftrightarrow B^1_\omega(n)$ holds under a weak arithemetic.*

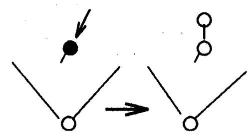
Proof. $\Rightarrow$ is from Lemma 2 and $\Leftarrow$ is from Lemma 3. □
Hence from Proposition 2 and Lemma 5 the following proposition holds.

Proposition 4 *For an arbitrary natural number n,*

$$PA \vdash B^1_\omega(n)$$

Until the above we observe that B^1_ω-Game is equivalent to Kirby-Paris' Hydra Game. Then when it comes to $B^1_{\omega+1}$-Game which is a extension of B^1_ω-Game, what is a corresponding natural extension of Extended Kirby-Paris' Hydra Game into $\mathcal{T}_2$? The answer is following. We define Hydra Game over $\mathcal{T}_2$ as follows: If the color of the top node Hercules chops off is *white* then a tree transforms itself in the same manner as Extended Kirby-Paris' Hydra Game (Definition 8). If the color of the top node Hercules chops off is *black* then a tree transforms itself in the manner as the following figure; the transformed tree results from a given tree by changing the color of the node Hercules chops off into white and by sprouting only one new white head.

It is trivial that the above game is embeddable into $B^1_{\omega+1}$-Game. To the contrary, it is



easily checked by using D-Rule with respect to the above game that the reduction relations defined in the above game is strong enough to prove the well-orderness of $\prec_{B^1_{\omega+1}}$.

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